UNIVERSIDADE DE BRASÍLIA

INSTITUTO DE FÍSICA

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ON THE LIMITS OF THE PERTURBATIVE APPROACH IN QFT: A STUDY IN ALGEBRAIC QUANTUM FIELD THEORY

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Monografia apresentada ao Instituto de Física da Universidade de Brasília como parte dos requisitos necessários à obtenção do título de Bacharel em Física.

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Abstract

We investigate some of the major achievements and limitations of the perturbative formalism in QFT and compare with alternative formulations and solutions in Algebraic Quantum Field Theory that can lead to a more consistent theory of quantum fields.

Key-words: Quantum Field Theory. Mathematical Physics. Algebraic Quantum Field Theory.

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Introduction

The success of the perturbative approach in QFT cannot be underestimated. Calculations in QED reached astounding precision in the last century, with the predicted value of the magnetic dipole of the muon, for example, sharing six decimal places with the value experimentally measured:

$$(g_\mu - 2)_{theor.} = 233183478(308) \cdot 10^{-11}$$

$$(g_\mu - 2)_{exp.} = 233184600(1680) \cdot 10^{-11}$$

The same degree of precision however cannot be reached in QCD yet. Nevertheless, the theory has been shown to be free of any ultraviolet divergences and to also contain no free parameters. In addition to that, asymptotic freedom implies that at high energies it becomes simple and perturbation theory is a good enough approximation.

Historically though, the process of renormalizaton for removal of divergences has often been criticized. Dirac is famously quoted as having said that:

> "...this so-called good theory does involve neglecting infinities which appear in its equations, ignoring them in an arbitrary way. This is just not sensible mathematics. Sensible mathematics involves disregarding a quantity when it is small – not neglecting it just because it is infinitely great and you do not want it!"

hinting that the current perturbative paradigm cannot be the final theory to describe particles and fields. The perturbative series also has its limits when it comes to certain aspects of QCD such as infrared divergences and confinement. Moreover, in the years to come, many more mathematical inconsistencies would be found in the foundations of perturbative QFT, one of the most important ones being Haag's theorem, which led some to pursue a different line of reasoning and attempt to derive QFT from rigorous axioms and first principles, in what would become known as axiomatic quantum field theory. The developments in this area eventually led to the creation of the flourishing field of algebraic quantum field theory where ideas such as constructive quantum field theory and chiral CFT provide a way to tap into the problematic nonperturbative sector of QFT.

The aim of this thesis is to provide a general introduction to nonperturbative QFT, providing the main motivations and alternatives to solve the current challenges the standard perturbative approach faces. We begin in chapter 01 with a review of standard perturbative QFT, giving special attention to the renormalization procedure. We then discuss the mathematical inconsistencies that lie in the foundations of the perturbative approach in chapter 02 and conclude, in chapter 03, with an overview of algebraic quantum field theory and how it provides a solid framework for investigating the problems introduced before.

1 Perturbative Quantum Field Theory

1.1 Canonical Quantization

In quantum field theory one deals with field variables $\phi(x,t)$ which translate physical systems with an infinite number of degrees of freedom. For continuous systems like these the Lagrangian can be expressed as:

$$L(t) = \int d^3x \, \mathcal{L}(\phi_a, \partial_\mu \phi_a), \tag{1.1}$$

where $\mathcal{L}$ is the Lagrangian field density. The action then takes the form:

$$S = \int dt \int d^3x \, \mathcal{L}(\phi_a, \partial_\mu \phi_a)$$

$$S = \int d^4x \, \mathcal{L}(\phi_a, \partial_\mu \phi_a), \tag{1.2}$$

and the variation, requiring $\delta S = 0$ leads to the Euler-Lagrange equations for fields:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \, . \tag{1.3}$$

The development of Quantum Field Theory has been intrinsically connected to the development of relativistic quantum mechanics. The first attempt at formulating a relativistic theory of quantum mechanics was realized in the form of the Klein-Gordon equation:

$$(\partial^\mu \partial_\mu + m^2)\phi = 0, \tag{1.4}$$

derived from the Lagrangian density:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 \, . \tag{1.5}$$

It was noted however, as early as 1925 by Klein, that the solutions of the equation led to negative probabilities densities and therefore the usual wavefunction interpretation had to be abandoned. If we expand the variable ϕ as:

$$\phi(\mathbf{x}, t) = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}} \phi(\mathbf{p}, t), \tag{1.6}$$

we can rewrite the KG equation in Fourier space as:

$$\left[\frac{\partial^2}{\partial t^2} + (|\mathbf{p}|^2 + m^2) \right] \phi(\mathbf{p}, t) = 0. \tag{1.7}$$

This equation is analogous to the harmonic oscillator equation of motion, with frequency given by:

$$\omega_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}\ . \tag{1.8}$$

From QM we know that to solve a problem as such, given by the Hamiltonian:

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2\phi^2, \tag{1.9}$$

it is appropriate to write the variables in terms of the so called ladder operators a and $a^\dagger$:

$$\phi = \frac{1}{\sqrt{2\omega}}(a + a^\dagger),$$

$$p = -i\sqrt{\frac{\omega}{2}}(a - a^\dagger), \tag{1.10}$$

which in the continuous case, satisfy:

$$[a_{\mathbf{p}}, a^\dagger_{\mathbf{p'}}] = (2\pi)^3\delta^{(3)}(\mathbf{p} - \mathbf{p'}), \tag{1.11}$$

and whose corresponding action is shown to create or annihilate particles with energy $\omega_{\mathbf{p}}$:

$$|\mathbf{p}\rangle = \sqrt{2\omega_{\mathbf{p}}}a^\dagger_{\mathbf{p}}|0\rangle, \tag{1.12}$$

provided the normalization condition to ensure the Lorentz invariance of the theory (PESKIN; SCHROEDER, 1995):

$$\langle\mathbf{p}|\mathbf{q}\rangle = 2\omega_{\mathbf{p}}(2\pi)^3\delta^{(3)}(\mathbf{p} - \mathbf{q}). \tag{1.13}$$

Therefore, the solution in position space can be written as:

$$\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}(a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} + a^\dagger_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}). \tag{1.14}$$

That is, a sum of oscillation modes which is formally understood as the field. This provides us with an alternative interpretation for ϕ. In a similar fashion we can rewrite the momentum as:

$$\pi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}(-i)\sqrt{\frac{\omega_{\mathbf{p}}}{2}}(a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} - a^\dagger_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}). \tag{1.15}$$

And one can verify the following commutation relations:

$$[\phi(\mathbf{x}), \pi(\mathbf{x'})] = i\delta^{(3)}(\mathbf{x} - \mathbf{x'}). \tag{1.16}$$

Promoting the field ϕ and momentum π variables to operators and imposing the commutation relations are at the root of the quantization procedure. The same procedure holds for vector or tensor fields, one promotes the field variables to operators and then impose suitable commutation relations. We further illustrate this for the case of the Dirac equation:

$$i\gamma^\mu\partial_\mu\psi - m\psi = 0, \tag{1.17}$$

which arises from the following Lagrangian density:

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi. \tag{1.18}$$

The field variables can be expanded as:

$$\psi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_s (a^s_{\mathbf{p}} u^s(\mathbf{p}) e^{-i\mathbf{p}\cdot\mathbf{x}} + b^{s\dagger}_{\mathbf{p}} v^s(\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{x}})$$

$$\bar{\psi}(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_s (b^s_{\mathbf{p}} \bar{v}^s(\mathbf{p}) e^{-i\mathbf{p}\cdot\mathbf{x}} + a^{s\dagger}_{\mathbf{p}} \bar{u}^s(\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{x}}). \tag{1.19}$$

This time the creation and annihilation operators must obey anticommutation relations in order to preserve fermion statistics:

$$\{a^r_{\mathbf{p}}, a^{s\dagger}_{\mathbf{q}}\} = \{b^r_{\mathbf{p}}, b^{s\dagger}_{\mathbf{q}}\} = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\delta^{rs}. \tag{1.20}$$

And the relation between ψ and $\psi^\dagger$ is as follows:

$$\{\psi_a(\mathbf{x}), \psi^\dagger_b(\mathbf{y})\} = \delta^{(3)}(\mathbf{x}-\mathbf{y})\delta_{ab}. \tag{1.21}$$

The quantization procedure for nonabelian Yang Mills theories is more easily realized within the path integral formalism, which we will introduce in the upcoming sections.

1.2 The Interaction Picture

In QM one often talks of the Schrödinger picture or the Heisenberg picture. In the Schrödinger picture states evolve in time according to the equation:

$$i\hbar\frac{d|\psi\rangle_S}{dt} = H|\psi\rangle_S, \tag{1.22}$$

and operators are fixed. In the Heisenberg picture we have the opposite, states are fixed and operators evolve in time according to the equation:

$$\frac{d}{dt}O_H(t) = \frac{i}{\hbar}[H_H, O_H(t)] + \left(\frac{\partial O_S}{\partial t}\right)_H. \tag{1.23}$$

It's easy to show that these two pictures are related via a unitary transformation of the form:

$$O_H(t) = e^{iHt}O_S e^{-iHt}$$

$$|\psi\rangle_H = e^{iHt}|\psi\rangle_S. \tag{1.24}$$

In the interaction picture, also called the Dirac picture, both states and operators evolve in time. This is the most useful picture to work with in QFT. The Hamiltonian is split into two, with H_0 governing the evolution of operators and H_{int} the evolution of states.

$$H = H_0 + H_{int}. \tag{1.25}$$

The interaction part itself is time dependant, therefore:

$$H_{int} = e^{iH_0 t} H_{int} e^{-iH_0 t}. \tag{1.26}$$

And we reach the Schrödinger equation for evolution of states:

$$i\hbar \frac{d|\psi\rangle_S}{dt} = H|\psi\rangle_S$$

$$i\hbar \frac{d(e^{-iH_0 t}|\psi\rangle_I)}{dt} = (H_0 + H_{int})_S e^{-iH_0 t}|\psi\rangle_I$$

$$i\hbar \frac{d|\psi\rangle_I}{dt} = e^{iH_0 t}(H_{int})_S e^{-iH_0 t}|\psi\rangle_I$$

$$\Rightarrow i\hbar \frac{d|\psi\rangle_I}{dt} = H_I(t)|\psi\rangle_I. \tag{1.27}$$

1.3 The Perturbative Series

The solution of Eq. (1.27) should involve a unitary time evolution operator $U(t, t_0)$ so that:

$$|\psi(t)\rangle_I = U(t, t_0)|\psi(t_0)\rangle_I \tag{1.28}$$

And the problem is reduced to determining the operator satisfying:

$$i\frac{dU}{dt} = H_I(t)U \tag{1.29}$$

The solution is of the form $U = \exp(-iH_I t)$, or in power series form:

$$U(t, t_0) = 1 + (-i)\int_{t_0}^{t} dt_1 H_I(t_1) + \frac{(-i)^2}{2!}\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 T\{H_I(t_1)H_I(t_2)\} + ... \tag{1.30}$$

Where we introduced the time ordering operator, necessary for consistency considerations:

$$T\{O_1(t_1)O_2(t_2)\} = \begin{cases} O_1(t_1)O_2(t_2) \ t_1 > t_2 \\ O_2(t_2)O_1(t_1) \ t_2 > t_1 \end{cases} \tag{1.31}$$

For the second term in the series for instance we have:

$$\frac{1}{2}\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 T\{H_I(t_1)H_I(t_2)\} = \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H_I(t_1)H_I(t_2) \tag{1.32}$$

Which can be formally interpreted as a consequence of the symmetry property of the operator:

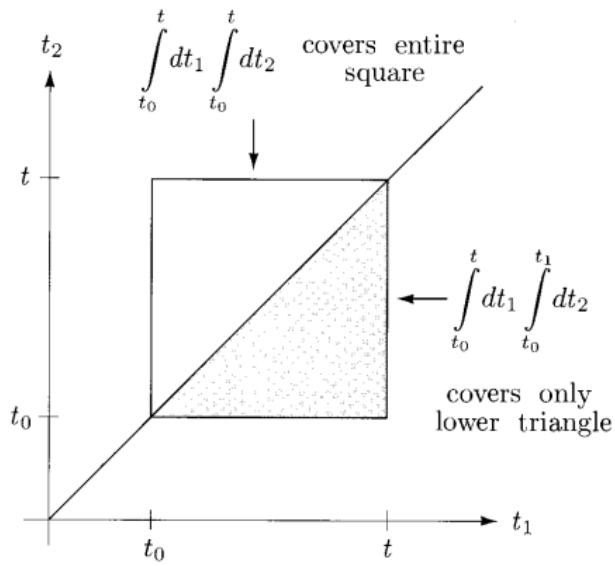


Figure 1 – Symmetry of time ordering operator. Ref. (PESKIN; SCHROEDER, 1995)

The perturbative series make it possible to calculate the fundamental objects in quantum field theory, the correlation functions, or Green functions $G^n(x_1, ..., x_n)$ involved in scattering processes. We'll develop the appropriate formalism in the next section, but for the moment we'll illustrate the calculation for the two point correlation function $\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle$. To achieve this, first we need to realize that the ground state of $H|\Omega\rangle$ has an overlap with the ground state of H_0, $|0\rangle$ so that:

$$e^{-iHT}|0\rangle = e^{-iE_0T}|\Omega\rangle\langle\Omega|0\rangle + \sum_{n\neq 0} e^{-iE_nT}|n\rangle\langle n|0\rangle. \tag{1.33}$$

Now, by the Riemann-Lebesgue lemma we have that:

$$\lim_{\mu\to\infty} \int_a^b dx f(x) e^{i\mu x} = 0. \tag{1.34}$$

So that the second term cancels of and the ground state Ω can be written as, after a small time shift:

$$|\Omega\rangle = (e^{-iE_0(t_0-(-T))}\langle\Omega|0\rangle)^{-1} U(t_0, -T)|0\rangle. \tag{1.35}$$

And the two point correlation function becomes:

$$\langle\Omega\phi(x)\phi(y)|\Omega\rangle = (|\langle 0|\Omega\rangle|^2 e^{-iE_0(2T)})^{-1} \times \langle 0|U(T, x^0)\phi_I(x)U(x^0, y^0)\phi_I(y)U(y^0, -T)|0\rangle. \tag{1.36}$$

Upon normalizing, dividing by $1 = \langle\Omega|\Omega\rangle$ we reach:

$$\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle = \frac{\langle 0|T\{\phi_I(x)\phi_I(y)\exp[-i\int_{-T}^{T} dt H_I(t)]\}|0\rangle}{\langle 0|T\{\exp[-i\int_{-T}^{T} dt H_I(t)]\}|0\rangle}, \tag{1.37}$$

as it will be shown the term in the denominator is precisely the expectation value of the S-Matrix in the vacuum.

1.4 The S-Matrix

In QFT we are ultimately interested in calculating scattering amplitudes, something which is achieved via the S-matrix. Assuming that the initial and final states of the scattering process are eigenstates of the free theory, one can write:

$$\lim_{t_\pm \to \pm\infty} \langle f|U(t_+, t_-)|i\rangle = \langle f|S|i\rangle, \tag{1.38}$$

where the S is the S-matrix unitary operator. The value of the expectation value should reflect momentum conservation, so that if we start with an incident particle beam composed of states with k_A, k_B momentum and end with states with $p_1, p_2, ...$ for momentum, in order to enforce momentum conversation we should have:

$$\langle p_1 p_2 ...|S|k_A k_B\rangle = (2\pi)^4 \delta^{(4)}(k_A + k_B - \sum p_f) \cdot iM(k_A, k_B \to p_f). \tag{1.39}$$

That is, we have a probability transition amplitude and the delta function translating the momentum conservation of the scattering process. Once in possession of the amplitude, it is easy to obtain the scattering cross section:

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{1}{2E_A E_B|v_A - v_B|} \frac{|p_1|}{(2\pi)^4 4E_{CM}} |M(p_A, p_B \to p_1, p_2)|^2, \tag{1.40}$$

and the decay rate:

$$d\Gamma = \frac{1}{2m_A}\left(\prod_f \frac{d^3 p_f}{(2\pi)^3}\frac{1}{2E_f}\right)|M(m_A \to p_f)|^2 (2\pi)^4 \delta^{(4)}(p_A - \sum p_f), \tag{1.41}$$

which are crucial for the detection and study of particle resonances at large particle accelerators.

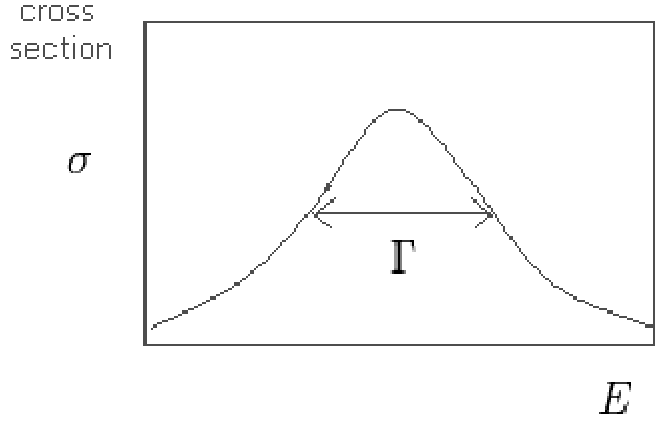


Figure 2 – Typical shape of a resonance signal at a particle detector.

The calculation of the transition amplitude is achieved via the comparison with the results in the perturbative series, where each term will give a partial amplitude contributing to the process.

$$S = \underbrace{I}_{S^{(0)}} \underbrace{-i\int_{-\infty}^{\infty} H_I(x_1)d^4x_1}_{S^{(1)}} \underbrace{-\frac{1}{2!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T\{H_I(x_1)H_I(x_2)\}d^4x_1d^4x_2}_{S^{(2)}} + ... \tag{1.42}$$

We saw how the perturbative approach allowed for the computation of the correlation functions, and how these are related to the S-Matrix via Eq. (1.38).

Since the terms in the perturbative series are reduced to the calculation of expressions of the form $\langle 0|T(\phi(x_1)\phi(x_2)....\phi(x_n)|0\rangle$, as in the two point correlation function, we will develop a method for calculating such values in this section.

Taking scalar theory as an example, we will start by decomposing our field solution (2.1.14) into:

$$\phi_I(x) = \phi_I^+(x) + \phi_I^-(x),$$

where:

$$\phi_I^+(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}} a_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}}$$

$$\phi_I^-(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}} a_{\mathbf{p}}^\dagger e^{+i\mathbf{p}\cdot\mathbf{x}}. \tag{1.43}$$

In this decomposition, the time ordered product $T\phi_I(x)\phi_I(y)$ will be given by:

$$T\phi_I(x)\phi_I(y) = \phi_I^+(x)\phi_I^+(y) + \phi_I^+(x)\phi_I^-(y) + \phi_I^-(x)\phi_I^+(y) + \phi_I^-(x)\phi_I^-(y). \tag{1.44}$$

It is useful however to introduce a convention called normal ordering of operators, which places all a_p terms, terms that have vanishing expectation vacuum expectation value, to the right, and therefore:

$$T\phi_I(x)\phi_I(y) = \phi_I^+(x)\phi_I^+(y) + \phi_I^-(y)\phi_I^+(x) + \phi_I^-(x)\phi_I^+(y) + \phi_I^-(x)\phi_I^-(y) + [\phi_I^+(x), \phi_I^-(y)]. \tag{1.45}$$

The relevant term here is the commutator, which defines what is called a contraction:

$$\contraction{}{\phi}{(x)}{\phi}\phi(x)\phi(y) = [\phi_I^+(x), \phi_I^-(y)] \text{ for } x^0 > y^0; [\phi_I^+(y), \phi_I^-(x)] \text{ for } y^0 > x^0. \tag{1.46}$$

This term has a natural interpretation which is that of a propagator. One can think of a particle that is created at one point and annihilated at the other.

$$\contraction{}{\phi}{(x)}{\phi}\phi(x)\phi(y) = D_F(x-y)$$

The evaluation in position space yields:

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)}. \tag{1.47}$$

We now rewrite the time ordered vacuum expectation value as:

$$T\{\phi(x)\phi(y)\} = N\{\phi(x)\phi(y) + \phi(x)\phi(x)\}. \tag{1.48}$$

The more general result for n fields is called Wick's theorem and states that any time ordering can be written as a normal ordering plus all possible contractions:

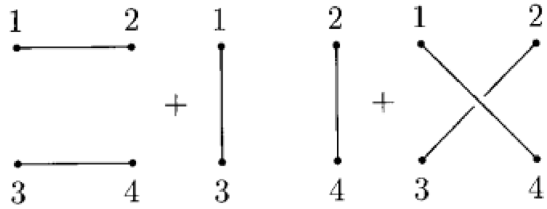
$$T\{\phi(x_1)\phi(x_2)...\phi(x_m)\} = N\{\phi(x_1)\phi(x_2)...\phi(x_m) + \text{contractions}\}. \tag{1.49}$$

Considering for instance an interaction theory such as Yukawa theory, whose interaction part of the Hamiltonian is given by:

$$H_I(x) = g\frac{\phi^4}{4}.$$

The linear part of the S-matrix then will involve terms such as:

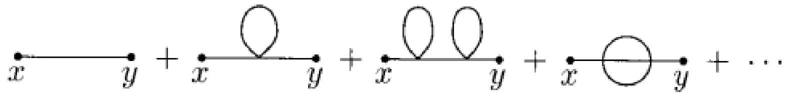
$$\langle 0| T\{\phi_1\phi_2\phi_3\phi_4\} |0\rangle =$$



And the higher order terms in the S-matrix expansion will involve even more complicated contractions, so that in general, the two point correlation function in terms of diagrams is:

$$\langle\Omega| T[\phi(x)\phi(y)] |\Omega\rangle$$
$= $ sum of all connected diagrams with two external points

$=$



Those are called Feynman diagrams.

One important aspect to keep into consideration is that only connected diagrams contribute to the amplitude. That is to say that even though the disconnected parts of the diagram below are possible results of a contraction, they aren't directly involved in the calculation of the correlation functions for scattering processes.

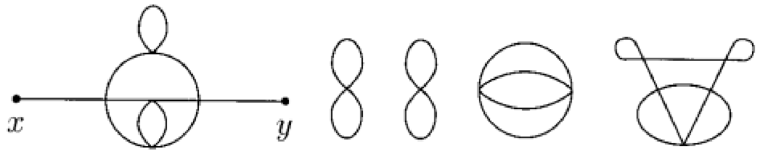


Figure 3 – Disconnected parts of the Feynman diagram on the left, known as vacuum bubbles.

This is related to the assumption we made earlier that the particle states during scattering are eigenstates of the free theory. In fact, it can be shown that the element in the denominator of Eq. (1.37) corresponds to the following sum of disconnected diagrams which one can interpret as transitions occurring in the vacuum:

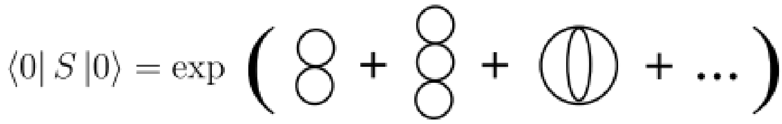


Figure 4 – Disconnected diagrams corresponding to vacuum to vacuum transitions.

As it turns out, the numerator term in Eq.(1.37) is such that:

$$\langle 0|T\phi_1...\phi_n S|0\rangle = (\sum \text{connected diagrams})\langle 0|S|0\rangle. \tag{1.50}$$

And so, dividing by $\langle 0|S|0\rangle$ we see that the two point correlation function derived earlier truly only depends on connected diagrams.

We now move on to a more complicated theory of interaction, namely quantum electrodynamics, or QED for short, for which the interaction part of the Hamiltonian is given by:

$$H_I(x) = -e\bar{\psi}\gamma^\mu A_\mu \psi. \tag{1.51}$$

Processes of interest described by this interaction Hamiltonian include for instance Compton scattering, associated to diagrams of this form in the expansion:

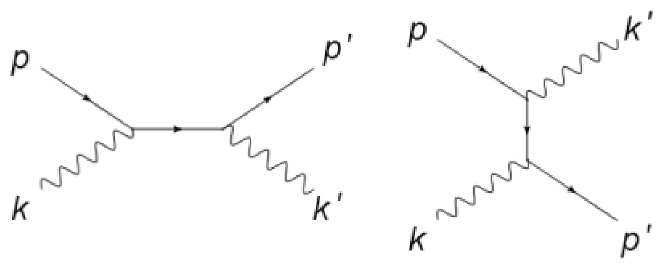


Figure 5 – 2-point functions involving the fermion and photon propagators, whose amplitude contributes to compton scattering.

One important result of this theory was the calculation of the anomalous magnetic moment of the electron, which we summarize below.

At the classical level the interaction between an electron and a magnetic field is given by the following Hamiltonian:

$$H = \frac{p^2}{2m} + V(r) + \frac{e}{2m} B \cdot (L + gS). \tag{1.52}$$

The g factor above measures the strength of the magnetic dipole interaction $eB \cdot S$. At around 1947 there was a disagreement between theory and experiment, with the value arising from experiments showing that it simply couldn't be an exact value like the one of $g = 2$ theoretically predicted. The real value of g can actually be deduced from QED diagrams. Consider for instance the tree level diagram below:

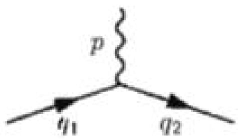
$i\mathcal{M}_0^\mu =$  $= -ie\bar{u}(q_2)\gamma^\mu u(q_1)$

Figure 6 – Tree level diagram for the process e$^-(q_1)A_\mu(p) \to e^-(q_2)$

Applying Gordon's identity:

$$\bar{u}(q_2)(q_1^\mu + q_2^\mu)u(q_1) = (2m)\bar{u}(q_2)\gamma^\mu u(q_1) + i\bar{u}(q_2)\sigma^{\mu\nu}(q_1^\nu - q_2^\nu)u(q_1). \tag{1.53}$$

We reach:

$$M_0^\mu = -e\left(\frac{q_1^\mu + q_2^\mu}{2m}\right)\bar{u}(q_2)u(q_1) - \frac{e}{2m} i\bar{u}(q_2)p_\nu\sigma^{\mu\nu}u(q_1). \tag{1.54}$$

The second term shows some spin dependence in the form of $\sigma^{\mu\nu}$ and is what is actually translating the magnetic moment at the quantum level. We can then identify g with the coefficient of the term $i\bar{u}(q_2)p_\nu\sigma^{\mu\nu}u(q_1)$ times $4m/e$, so that at the tree level we reach g=2. We are however interested in higher order corrections arising at the loop level that could possibly account for the experimental results.

It can be shown that the most general expression is of the form (RYDER, 1996):

$$iM^\mu = (-ie)\bar{u}(q_2)\left[F_1\left(\frac{p^2}{m^2}\right)\gamma^\mu + \frac{i\sigma^{\mu\nu}}{2m}p_\nu F_2\left(\frac{p^2}{m^2}\right)\right]. \tag{1.55}$$

At tree level for instance as we have seen we have $F_1 = 1$ and $F_2 = 0$.

The term that actually generates corrections for the magnetic moment at higher orders however is F_2 in the non relativistic regime, so that the problem is reduced to calculating:

$$g = 2 + 2F_2(0). \tag{1.56}$$

And from the calculation of the diagram below:

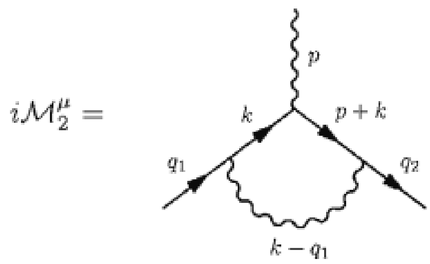


Figure 7 – 1-loop diagram contributing to the form factor F_2.

The value of $F_2(0)$ can be shown to be (RYDER, 1996):

$$F_2(0) = \frac{\alpha}{2\pi}. \tag{1.57}$$

So that we reach:

$$g = 2 + \frac{\alpha}{2\pi} = 2.00232. \tag{1.58}$$

This result was first achieved by Schwinger in 1948 and constitutes a major achievement of quantum field theory.

1.5 The Path Integral Formulation

The same concepts developed so far can be approached via the so-called path integral formulation. The main idea is to write the total amplitude for a particle propagating from x_a to x_b as a sum of amplitudes for all paths possible that differ only by a phase factor:

$$U(x_a, x_b; T) = \sum_{\text{all paths}} e^{i\delta} = \int \mathcal{D}x(t) e^{i\delta}. \tag{1.59}$$

In the picture on the next page, all the paths labeled from 1 to 4 and differing only by certain phases should contribute to the amplitude of the particle to go from point a to b.

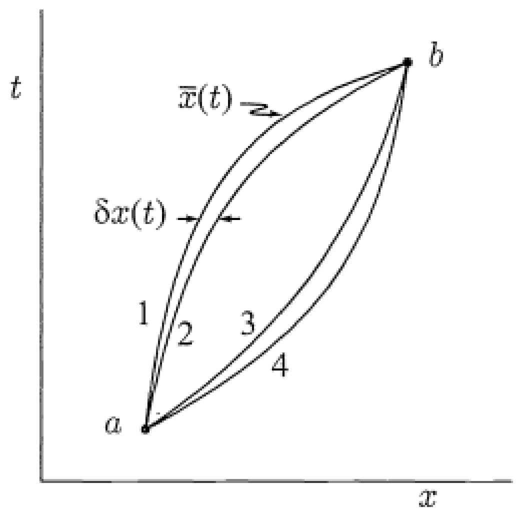


Figure 8 – In quantum mechanics, the amplitude to go from a to b should the sum of amplitudes for each interfering alternative path that differ from another by a phase. Ref.(FEYNMAN; HIBBS, 1965)

Taking the classical limit into consideration it makes sense to consider the action $S[x]$ as the phase in question, so that:

$$U(x_a, x_b; T) = \int \mathcal{D}x(t) e^{i\S[x(t)]/\hbar}. \tag{1.60}$$

In the case of a more general quantum system described by a set of coordinates q^i and p^i the transition amplitude takes the following form:

$$U(q_a, q_b; T) = \left(\prod_i \int \mathcal{D}q(t)\mathcal{D}p(t)\right) \exp[i \int_0^T dt(\sum_i p^i \dot{q}^i - H(q^i, p^i))]. \tag{1.61}$$

This will allow us to write transition functions for quantum fields. In the scalar theory case for instance it can be shown that the two point correlation function derived as an example in Eq. (1.37), can be written as:

$$\langle \Omega | T\phi(x_1)\phi(x_2) | \Omega \rangle = \frac{\int \mathcal{D}\phi \phi(x_1)\phi(x_2) \exp[i \int_{-T}^{T} d^4x \mathcal{L}]}{\mathcal{D}\phi \exp[i \int_{-T}^{T} d^4x \mathcal{L}]}. \tag{1.62}$$

In general however the correlation functions are obtained by what is called a generating functional, which is analogous to the partition function in statistical mechanics.

$$Z[J] = \int \mathcal{D}\phi \exp[i \int d^4x(\mathcal{L} + J\phi)]. \tag{1.63}$$

In terms of the Feynman propagator, it can be written as:

$$Z[J] = Z_0 \exp\left[-\frac{i}{2} \int J(x)\Delta_F(x-y)J(y)d_x d_y\right], \tag{1.64}$$

where Z_0 corresponds to Z[J=0]. For ϕ^4 theory for instance, we have:

$$Z[J] = \frac{\exp\left[i\int \mathcal{L}_{\text{int}}\left(\frac{1}{i}\frac{\delta}{\delta J(z)}\right)dz\right]\exp[-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)dx\,dy]}{\left\{\exp\left[i\int \mathcal{L}_{\text{int}}\left(\frac{1}{i}\frac{\delta}{\delta J(z)}\right)dz\right]\exp[-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)dx\,dy]\right\}\Big|_{J=0}}. \tag{1.65}$$

Expanding the numerator term in the expression above in g to first order gives:

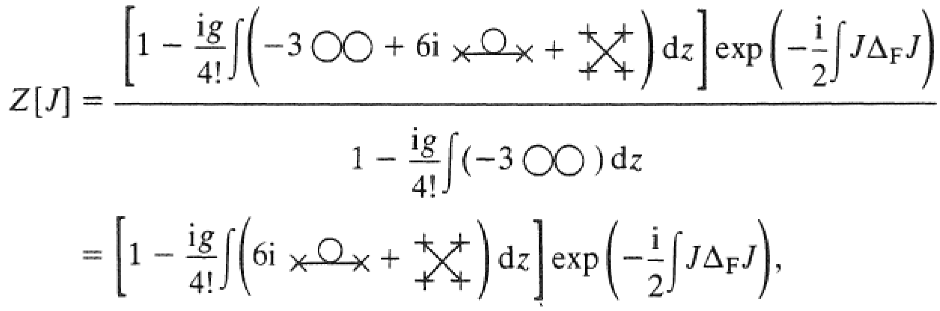
$$\left[1-\frac{ig}{4!}\int\left(\frac{1}{i}\frac{\delta}{\delta J(z)}\right)^4 dz + O(g^2)\right]\exp[-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)dx\,dy]. \tag{1.66}$$

Which corresponds to:

$$\left\{-3[\Delta_F(0)]^2 + 6i\Delta_F(0)\left[\int\Delta_F(z-x)J(x)dx\right]^2 + [\Delta_F(z-x)J(x)dx]^4\right\}, \tag{1.67}$$

and $\Delta_F(0) = \Delta_F(x-x)$ corresponds to the closed loop:

So we see that:



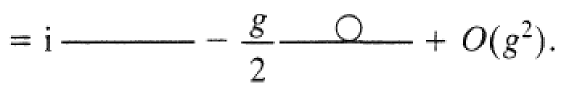
This gives us some insight on how the generating functional can be used to obtain correlation functions. The two point correlation function for instance, corresponds to the term:

$$\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = Z[J]^{-1}\left(-i\frac{\delta}{\delta J(x_1)}\right)\left(-i\frac{\delta}{\delta J(x_2)}\right)Z[J]|_{J=0}, \tag{1.68}$$

which in terms of the propagators gives:

$$= i\Delta_F(x_1-x_2) - \frac{g}{2}\Delta_F(0)\int dz\Delta_F(z-x_1)\Delta_F(z-x_2) + O(g^2), \tag{1.69}$$

and as graphs:



In general, denoting the n-point correlation function by $\tau(x_1, ...x_n)$, we have:

$$\tau(x_1, ...x_n) = \frac{1}{i^n} \frac{\delta Z[J]}{\delta J(x_1)...\delta J(x_n)}\Big|_{J=0} \tag{1.70}$$

Note how the terms in the generating functional expansion at our example above contain disconnected diagrams, this constitutes a problem since as shown previously these diagrams correspond to vacuum to vacuum transitions and don't really interfere in the scattering process. What we are interested then is a generating functional for connected diagrams. It can be shown that such functional is related to Z[J] via (RYDER, 1996):

$$W[J] = -i \ln Z[J]. \tag{1.71}$$

And the functional derivation of this object defines the so called irreducible n-point functions:

$$\phi(x_1, ...x_n) = \frac{1}{i^n} \frac{\delta^n W[J]}{\delta J(x_1)...\delta J(x_n)}\Big|_{J=0}. \tag{1.72}$$

which only gives rise to connected Feynman diagrams. A third object worth mentioning is the effective action Γ function, defined in terms of the Legendre transform:

$$W[J] = \Gamma[\phi] + \int dx J(x)\phi(x), \tag{1.73}$$

it can be shown that it corresponds to exactly the inverse of the n-point function, but more importantly it only generates so called 1PI-irreducible diagrams, that is amputated diagrams without external leg corrections (RYDER, 1996).

The formalism developed so far also holds for spinor fields, but one needs to take into account that those follow fermion statistics, so that it is necessary to write the field variables as:

$$\psi(x) = \sum_i \psi_i \phi_i(x), \tag{1.74}$$

where $\phi_i(x)$ are basis functions and ψ_i are anticommuting Grassmann numbers. The two point function then can be written as (PESKIN; SCHROEDER, 1995):

$$\langle\Omega|T\psi(x_1)\bar{\psi}(x_2)|\Omega\rangle = \frac{\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\ \psi(x_1)\bar{\psi}(x_2)\exp[i\int_{-T}^{T} d^4x\bar{\psi}(i\partial - m)\psi]}{\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\exp[i\int_{-T}^{T} d^4x\bar{\psi}(i\partial - m)\psi]}, \tag{1.75}$$

and the generating functional as a function of Grassmann-valued source fields $\eta(x)$, $\bar{\eta(x)}$ is:

$$Z[\eta, \bar{\eta}] = Z_0 \exp[-\int d^4x d^4y \bar{\psi}(i\partial - m)\psi + \bar{\eta}\psi + \bar{\psi}\eta]. \tag{1.76}$$

Analogously, one calculates correlation functions via functional derivation:

$$\langle 0|T\psi(x_1)\bar{\psi(x_2)}|0\rangle = Z_0^{-1}\left(-i\frac{\delta}{\delta\bar{\eta}(x_1)}\right)\left(+i\frac{\delta}{\delta\eta(x_2)}\right) Z[\eta, \bar{\eta}]|_{\eta, e\bar{t}a=0}. \tag{1.77}$$

In the case of abelian gauge theories such as QED, the gauge must be fixed during quantization in what is known as the Faddeev-Popov procedure. Consider the action for the EM field:

$$S = \int d^4x[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}]$$

$$S = \frac{1}{2}\int d^4x \frac{1}{2}A_\mu(x)(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu)A_\nu(x). \tag{1.78}$$

Gauge freedom of the form:

$$A'_\mu = A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x), \tag{1.79}$$

implies that if one were to take the class of solutions corresponding to $A'_\mu(x) = 0$, that is $A_\mu(x) = \frac{1}{e}\partial_\mu\alpha(x)$ the action would vanish and the functional integral $\int \mathcal{D}Ae^{iS[A]}$ would be divergent.This problem arises from the redundancy of summing over all field configurations that are gauge equivalent. In order to solve this one must fix the gauge. Let us adopt the Lorentz gauge condition $G(A) = 0$, where:

$$G(A) = \partial_\mu A^\mu. \tag{1.80}$$

This constraint is set in the functional integral in the form of a functional delta function $\delta(G(A))$ in the term:

$$1 = \int \mathcal{D}\alpha(x)\delta(G(A^\alpha)) \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right). \tag{1.81}$$

So that the transition function is given by:

$$\det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right)\int \mathcal{D}\alpha(x)\int \mathcal{D}Ae^{iS[A]}\delta(G(A^\alpha)). \tag{1.82}$$

This procedure of gauge fixing carries over to non-abelian gauge theories, however the determinant term in this case is dependant on the field and therefore will introduce new degrees of freedom to the theory associated to what are called Faddeev-Popov ghosts (PESKIN; SCHROEDER, 1995):

$$\det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right) = \int \mathcal{D}c\mathcal{D}\bar{c}\exp[i\int d^4x\bar{c}(-\partial^\mu D_\mu)c] \tag{1.83}$$

One then is able to quantize all theories discussed so far and even calculate correlation functions via the generating functional. It is important to mention however that the path integral formalism with all its advantages is not completely equivalent to the usual formulation derived previously and as we will see in Ch.02 the general intuition that is behind the path integral as it was presented lacks a solid mathematical formulation in respect to what exactly is meant by summing over all paths and so on.

1.6 Renormalization

In the expansion presented in the previous section for scalar theory we encountered terms of the form:

$$g \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 - m^2}, \tag{1.84}$$

corresponding to the loop diagram:

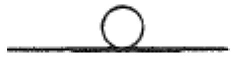


Figure 9 – 2-point loop function

And in second order even:

$$g^2 \int \frac{d^4p}{(2\pi)^8} \frac{1}{(p^2 - m^2)[(p_1 + p_2 - p)^2 - m^2}, \tag{1.85}$$

corresponding to the following loop diagram:

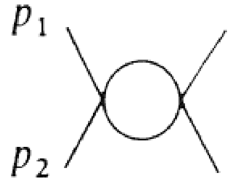


Figure 10 – 4-point loop function

It is clear that these diagrams produce divergent results. Simple power counting of the momentum shows that the integral corresponding to the first one is a case of quadratic divergence whereas the second one diverges logarithmically. The procedure for making terms such as these finite is called renormalization and is a key aspect of QFT. It can be summarized in three steps as follows. First the integral is regularized, which is to say that is made convergent via special mathematical techniques and then the characteristic energy scale, called the cutoff limit, is set. Once this is done, one needs to redefine the coupling in the expansion, a procedure that can be carried out by slightly modifying the original Lagrangian. If this step can be carried out, the theory is said to be renormalizable (COLLINS, 1984). The final step is to remove the cutoff limit, making the theory independent of the energy scale.

Let us illustrate each step in the case of scalar theory. The integral at Eq. (1.84) can be made convergent via a trick called dimensional regularization. Integrating the expression in d dimensions and multiplying by a factor μ^{4-d} for dimensional considerations, we have:

$$\frac{1}{2}g\mu^{4-d}\int\frac{d^dp}{(2\pi)^d}\frac{1}{p^2-m^2}. \tag{1.86}$$

To evaluate the integral we make use of the general result provided in (RYDER, 1996):

$$\int\frac{d^dp}{(2\pi)^d}\frac{1}{(p^2+\Delta)^n}=\frac{1}{(4\pi)^{d/2}}\frac{\Gamma(n-\frac{d}{2})}{\Gamma(n)}(\frac{1}{\Delta})^{n-\frac{d}{2}}, \tag{1.87}$$

reaching:

$$-\frac{ig}{32\pi^2}m^2\left(\frac{4\pi\mu^3}{-m^2}\right)^{2-d/2}\Gamma\left(1-\frac{d}{2}\right) \tag{1.88}$$

Now, defining $\epsilon = 4-d$, one can use the expansion:

$$\Gamma(1-d/2)=\Gamma(-1+\epsilon/2)=-\frac{2}{\epsilon}-1+\gamma+O(\epsilon), \tag{1.89}$$

and the approximation $a^\epsilon = 1+\epsilon\ln a$, to reach the regularized form of the integral:

$$\frac{igm^2}{16\pi^2\epsilon}+\frac{igm^2}{32\pi^2}\left[1-\gamma+\ln\left(\frac{4\pi\mu^2}{-m^2}\right)\right]+O(\epsilon). \tag{1.90}$$

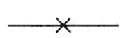
Similar regularization procedures can be carried out for the 4-point function where it can be shown that the integral at Eq. (1.85) reduces to (RYDER, 1996):

$$\frac{ig^2\mu^\epsilon}{16\pi^2\epsilon}+\text{finite}. \tag{1.91}$$

These expression however are still divergent as $\epsilon \to 0$. To fix this at the level of the Lagrangian we must add counter terms. For the 2-point function such term is:

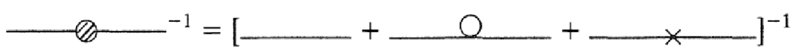
$$\delta\mathcal{L}_1=-\frac{gm^2}{32\pi^2\epsilon}\phi^2=-\frac{\delta m^2}{2}\phi^2. \tag{1.92}$$

Note that this term has the effect of introducing an interaction that effectively gets rid of the divergent term:



$$= -\frac{igm^2}{16\pi^2\varepsilon} = -i\delta m^2.$$

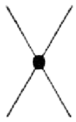
$\Rightarrow$



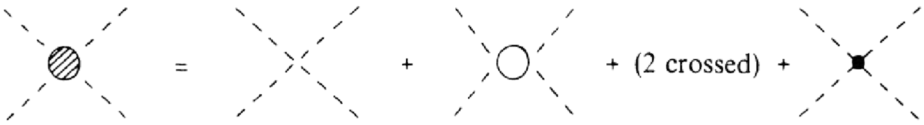
For the four point function the counter term added to the Lagrangian should be:

$$\delta\mathcal{L}_2=-\frac{1}{4!}\frac{3g^2\mu^\epsilon}{16\pi^2\epsilon}\phi^4, \tag{1.93}$$

since it gives rise to the Feynman rule:



$\Rightarrow$



Higher order terms also require counter terms such as:

$$\delta\mathcal{L}_3 = \frac{A}{2}(\partial_\mu\phi)^2. \tag{1.94}$$

So that the Lagrangian now becomes:

$$\mathcal{L}_B = \left(\frac{1+A}{2}\right)(\partial_\mu\phi)^2 - \frac{(m^2+\delta m^2)}{2}\phi^2 - (1+B)\frac{g\mu^\epsilon}{4!}\phi^4, \tag{1.95}$$

and with the following redefinitions:

$$\phi_B = \sqrt{Z_\phi}\phi, \qquad Z_\phi = 1 + A,$$

$$m_B = Z_m m, \qquad Z_m^2 = \frac{m^2+\delta m^2}{m^2(1+A)},$$

$$g_B = \mu^g Z_g g, \qquad Z_g = \frac{1+B}{(1+A)^2}.$$

We reach what is called the bare Lagrangian:

$$\mathcal{L}_B = \frac{1}{2}(\partial_\mu\phi_B)^2 - \frac{m_B^2}{2}\phi^2 - \frac{g_B}{4!}\phi^4. \tag{1.96}$$

This bare Lagrangian is what actually constitutes the physical theory being tested in QFT experiments.

An important consequence of renormalization is that the renormalized functions must remain invariant under change of a regularisation parameter μ. This result is expressed via the so called renormalization group equation:

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - n\gamma(g) + m\gamma_m(g)\frac{\partial}{\partial m}\right]\Gamma^{(n)} = 0, \tag{1.97}$$

where:

$$\gamma(g) = \mu \frac{\partial}{\partial \mu} \ln \sqrt{Z_\phi},$$

$$\beta(g) = \mu \frac{\partial g}{\partial \mu},$$

$$m\gamma_m(g) = \mu \frac{\partial m}{\partial \mu}.$$

A lot can be deduced about a theory from these expressions. In particular one is usually interested in the asymptotic behaviour of the $\beta(g)$ function above. At the fixed point $\beta(g) = 0$ the theory is scale invariant since $g =$ const. and analyzing the behaviour around that we can judge the validity of the perturbative expansion. For $\beta(g) > 0$ the coupling increases with the energy and the perturbative expansion becomes progressively worse.

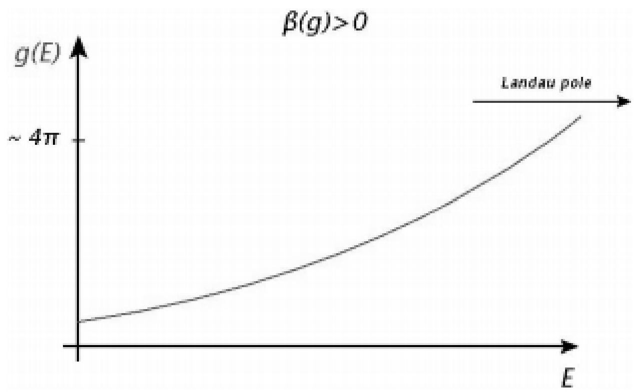


Figure 11 – Evolution of the coupling with energy for a positive beta function.

On the other hand, for $\beta(g) < 0$, the coupling decreases, effectively vanishing at high energies. In this situation the perturbative expansion becomes better and better and is what is known as asymptotic freedom:

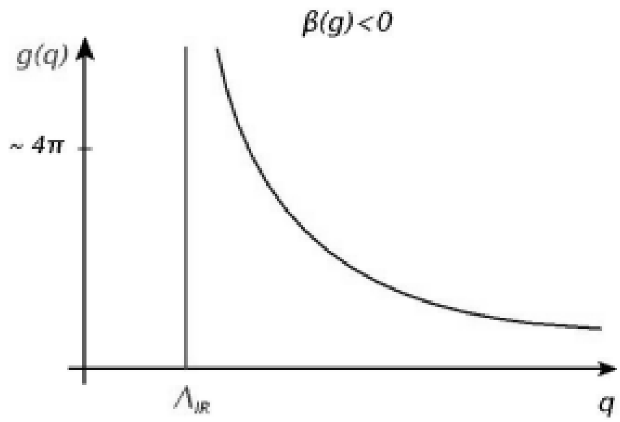


Figure 12 – Evolution of the coupling with energy for a negative beta function.

Let us verify the asymptotic behaviour of scalar theory. We have (RYDER, 1996):

$$g_B = g\mu^{\epsilon}\left(1 + \frac{3g}{16\pi^2\epsilon}\right). \tag{1.98}$$

Therefore:

$$\mu\frac{\partial g}{\partial \mu} = \epsilon g\mu^{\epsilon} + \frac{3g^2}{16\pi^2}\mu^{\epsilon}. \tag{1.99}$$

In the limit $\epsilon \to 0$, we reach the solution:

$$g = \frac{g_0}{1 - \frac{3}{16\pi^2}g_0 \ln\left(\frac{\mu}{\mu_0}\right)}. \tag{1.100}$$

So g increases with μ and the theory is not asymptotically free.

In the case of QED, the calculation of the divergent loop diagrams pictured below is quite involved and will not be carried thoroughly here.

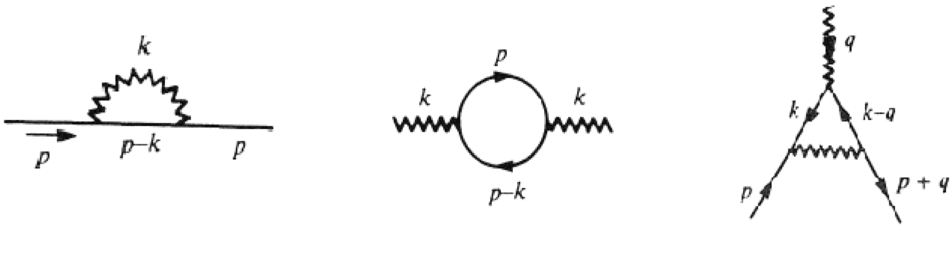


Figure 13 – Divergent diagrams in QED: the electron and photon self energy along the vertex function. Ref.(RYDER, 1996)

It is however interesting to investigate the asymptotic behavior of the theory as it was done for the scalar case. In QED it can be shown that the coupling varies according to (RYDER, 1996):

$$e_B = e\mu^{\epsilon/2}\left(1 + \frac{e^2}{12\pi^2\epsilon}\right). \tag{1.101}$$

In the limit $\epsilon \to 0$, the beta function becomes:

$$\mu\frac{\partial e}{\partial \mu} = \frac{e^3}{12\pi^2}, \tag{1.102}$$

and as in the scalar theory case, the theory shows an increasing coupling constant with increasing scale, so that it is also not asymptotically free:

$$e^2(\mu) = \frac{e^2(\mu_0)}{1 - \frac{e^2(\mu_0)}{6\pi^2}\ln\frac{\mu}{\mu_0}}. \tag{1.103}$$

The singularity at:

$$\mu = \mu_0 \exp\left(\frac{6\pi^2}{e^2(\mu_0)}\right) \tag{1.104}$$

is called Landau pole and is particularly problematic since at this point the coupling constant becomes infinite. QED is therefore no good at larger momenta. This is not to say that the theory is not renormalizable however.

What about QCD? Surprisingly enough the bare coupling in QCD follows the form (RYDER, 1996):

$$g_B = g\mu^{\epsilon/2}\left[1 + \frac{g^2}{16\pi^2\epsilon}\left(-11 + \frac{2n_F}{3}\right)\right], \tag{1.105}$$

where n_F stands for the number of quark flavours. The beta function in the limit $\epsilon \to 0$ is:

$$\beta(g) = \frac{g^3}{16\pi^2}\left(-11 + \frac{2n_F}{3}\right). \tag{1.106}$$

Therefore, since $n_F < 16$, we have $\beta < 0$, that is a decreasing beta function and g decreases with the energy. Perturbation theory in this case is then a good enough approximation and the theory is said to be asymptotically free. What at first glance seems like an advantage actually produces a limitation when it comes to lower energy processes. The theory at this scale, called the confining scale, can no longer be adequately studied since, noting again at Fig. 12, it is plagued by infrared divergences. We shall return to this point in chapters 03 and 04.

2 Problems of the Perturbative Approach

2.1 Mathematical Rigour and Inconsistencies

The problems lying at the foundations of QFT begin with the canonical quantization procedure. For starters, $\phi(x)$ is not a well defined operator, since although the vacuum expectation value $\langle 0|\phi(x)|0\rangle = 0$, we have a divergent value of $\phi^2(x)$ in the form of $\langle 0|\phi(x)\phi(x)|0\rangle = \delta(x-x) = \delta(0) = \infty$. Which proves quantum fields defined in the manner of QFT possess intrinsic short-distance singularities (STROCCHI, 2013). Moreover, if fields are to be well defined mathematical objects, they need to be regarded as operator valued distributions, requiring smearing with a test function $f(x)$:

$$\phi(x) = \int d^4x f(x)\phi(x). \tag{2.1}$$

In the interacting case, things are even more complicated and smearing is not enough to guarantee the well-definiteness of the operators in question. Operations involving multiplying fields at a space time point are often undertaken without any regard to the fact that they are distributions and must satisfy certain restraints if such a operation is to be possible.

Something already mentioned about S-matrix theory is the assumption that the eigenstates of the free theory are the same as the ones in the interacting theory.

$$|\psi_{\text{int}}\rangle \propto \lim_{\epsilon\to 0} S_\epsilon(0,\pm\infty)|\psi_{\text{free}}\rangle. \tag{2.2}$$

This is known as adiabatic switching and has the effect of switching off the interaction outside of a compact region of spacetime. Although physically questionable since actual interactions in physics are not at all "switched off" anywhere, it does provides a way to make the limit $t\to\infty$, $t_0\to-\infty$ well defined.

$$S = \lim_{\epsilon\to 0}\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{\infty} dt_1 ... \int_{-\infty}^{\infty} dt_n e^{-\epsilon(|t_1|+...|t_n|)} T(H_I(t_1)...H_I(t_n)). \tag{2.3}$$

The original Neumann-Liouville expansion for the S-Matrix at Eq. (1.42) is in general divergent and requires a convergence factor. At the Hamiltonian level this is achieved by multiplying the interaction part by a smooth function g(x):

$$H = H_{\text{free}} + g(x)H_I\ . \tag{2.4}$$

Nevertheless, the adiabatic limit $g\to 1$, should be such that it allows the removal of the switching at the end. The problem arises when one realises the failure of this limit to exist because of infrared divergences (STROCCHI, 2013).

One might argue that the problem is solved after renormalization, but as show by Dyson that is not the case. Consider the renormalized perturbative expansion at the origin:

$$F(e^2) = F(0) + e^2 F_2(0) + e^4 F_4(0) + ... \tag{2.5}$$

The transition from positive to negative values should be easily achieved if the series in question is smooth and analytic, but as it turns out the physics described by $F(-e^2)$ is remarkably different from the positive one at Eq. (2.5) (STROCCHI, 2013). This puts into question the whole ethos of the renormalization procedure. It is difficult to reconcilliate the outstanding physical predictions with the non-analicity of the perturbative series.

These problems aren't absent in the path integral formalism either, where time ordering isn't present and there can very well be discontinuous or non-differentiable paths so that the sum is not mathematically rigorous. The problem then becomes one of defining an appropriate measure for this infinite dimensional space of paths. To this date, various approaches have been suggested (GLIMM; JAFFE, 2012)(CONNES; KREIMER, 2000)(HUANG; YAN, 2000), but the debate with mathematicians still lingers. We note also that the path integral formalism is only equivalent to canonical quantization in the euclidean formulation. The real Minkowskian QFT is expected to be obtained through analytic continuation on the euclidean time of the n-point functions.

2.2 Haag's Theorem

The main idea behind Haag's theorem can be summarized by saying that if a theory contains the vacuum as the only euclidean invariant state and the fields obey equal time canonical commutation relations, then the vacuums of the free Hamiltonian and the interaction theory are the same, thus implying a free theory. In conclusion, interaction theories are incompatible with standard properties of QFT. In order to sketch a proof that this is indeed the case, let us list the assumptions exactly and see how it turns out for the scalar case (STROCCHI, 2013):

i) the three-dimensional Euclidean group is implemented by unitary operators U(a, R), under which the fields transform covariantly, e.g., for a scalar field:

$$U(a, R)\phi(x, t)U(a, R) = \phi(Rx + a, t)$$

.

ii) the vacuum is the only Euclidean invariant state,

iii) the fields obey equal-time canonical (anti)commutation relations and their representation is a Fock irreducible representation, the vacuum state Ψ_0 and the Fock no-particle state Ψ_{0F} coincide.

Now consider the Fock no-particle state Ψ_{0F} obeying $a(f)\Psi_{0F} = 0$. We have:

$$a(f)U(a,R)^*\Psi_{0F} = U(a,R)^*a(f,R)\Psi_{0F} = 0, \tag{2.6}$$

and since $U(a,R)^*\Psi_{0F}$ obeys the Fock condition it must coincide with Ψ_{0F} apart from a phase factor. On the other hand, the phase factor provides a continuous representation of the euclidean group, but for this case in particular there is no other representation other than the trivial one, so that the phase must be $\omega = 1$. This implies that Ψ_{0F} is invariant under the euclidean group, but since by our assumption the vacuum of the free theory is the only euclidean invariant state, Ψ_{0F} must coincide with it.

In the more general version of Hall and Wightmann none of the fields are actually required to be free (FRASER, 2006). Let us consider two Hilbert spaces h_1 and h_2 each ecquiped with a representation of the euclidean group $D_j(a,R)$ $(j = 1,2)$.So that:

$$D_j(a,R)\phi_j(x)D_j(a,R)^\dagger = \phi_j(Rx+a). \tag{2.7}$$

Assuming there are invariant states $\phi_{0j} \in h_j$, such that $D_j\phi_{0j} = \phi_{0j}$ and considering the two theories are related via an unitary map $\phi_1(x) = V^{-1}\phi_2(x)V$, we have:

$$D_1(a,R) = V^{-1}D_2(a,R)V\ . \tag{2.8}$$

The map $V : h_1 \to h_2$ does not need to be trivial, however in the calculation of the n-point functions we have:

$$\begin{aligned}\langle\phi_{01}|\phi_1(x_1)...\phi_1(x_n)|\phi_{01}\rangle &= \langle V\phi_{01}|\phi_2(x_1)V...V^{-1}\phi_2(x_n)|V\phi_{01}\rangle \\ &= \langle\phi_{01}|V^{-1}\phi_2(x_1)V...V^{-1}\phi_2(x_n)V|\phi_{01}\rangle \\ &= \langle\phi_{02}|\phi_2(x_1)...\phi_2(x_n)|\phi_{02}\rangle.\end{aligned} \tag{2.9}$$

So the n-point functions coincide and we can't have a consistent theory of interaction. Haag's theorem can be circumvented by quantizing the theory within a finite volume region(KLACZYNSKI, 2016). By considering an expansion of the form:

$$\phi(x) = \frac{1}{\sqrt{V}}\sum_k \frac{1}{\sqrt{2E_k}}(a_k e^{-ikx} + a_k^\dagger e^{ikx}), \tag{2.10}$$

the vacuum overlapping problem is avoided. The obvious consequence however is the loss of the full Poincaré invariance of the theory.

2.3 QFT at High Energies

In the standard model particles acquire mass via the Higgs mechanism which requires a non vanishing vacuum expectation value for H at the minimum of the potential:

$$V = m_H^2|H|^2 + \lambda|H|^4. \tag{2.11}$$

So that $\langle H \rangle = \sqrt{-m_H^2/2\lambda}$ and given the experimental value of around $\langle H \rangle = 174$ GeV for the vacuum expectation value, we reach $\lambda = 0.126$ and $m_H^2 = -(92.9\text{ GeV})^2$. The problem however is that this particle receives corrections from every particle that it couples to in the SM Lagrangian.

Figure 14 – Corrections to the term m_H^2 due to the Higgs coupling to a fermion and to a scalar. Ref.(MARTIN, 1998)

The contribution from the first diagram in the picture above for instance, corresponding to the coupling of the Higgs with a fermion f yields the correction:

$$\Delta m_H^2 = -\frac{|\lambda_f|}{8\pi^2}\Lambda_{\text{UV}}^2 + ... \tag{2.12}$$

We see then that this term is proportional to the square of the momentum cutoff. The problem is that at high energies such as those around the Planck scale $M_p = 1/\sqrt{8\pi G} = 2,4 \cdot 10^{18}$ GeV this quantum correction to m_h^2 is 30 orders of magnitude larger than the required value of $m_H^2 \approx -(92.9\text{ GeV})^2$. It is immediate to note that this difficulty stems from the fact that the correction is quadratically divergent. Comparing this to the electron self energy in QED, we see just how the problem differs.

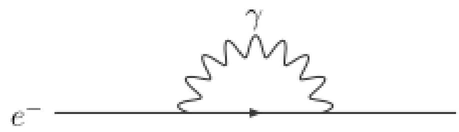


Figure 15 – Electron self energy in QED

Here the two point function is given by:

$$\begin{aligned}\pi_{ee}(0) &= \int \frac{d^4k}{(2\pi)^4}(-ie\gamma_\mu)\frac{i}{k - m_e}(-ie\gamma_\nu)\frac{-ig^{\mu\nu}}{k^2} \\ &= -e^2\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2(k^2 - m_e^2)}\gamma_\mu(k + m_e)\gamma^\mu \\ &= -4e^2 m_e\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2(k^2 - m_e^2)}.\end{aligned} \tag{2.13}$$

And even if we consider the integral evaluated at the Planck scale, we only obtain a modest correction, since it is only logarithmically divergent (DREES, 1996):

$$\Delta m_e \approx 2\frac{\alpha_{\mathrm{EM}}}{\pi} m_e \log \frac{M_p}{m_e} \approx 0.24 m_e \; . \tag{2.14}$$

The divergence of Δm_h^2 however is a much more serious problem since all the masses of the standard model are sensible to the vacuum expectation value. It also exemplifies a deep issue on how gravity and the weak force differ. The disparity between these values is what is known as the hierarchy problem and it tells us that there is likely new important physics happening at the Planck scale that can't be adequately described by the current perturbative paradigm.

2.4 QFT on Gravitational Backgrounds

The formal description of quantum fields on curved spacetimes only received proper attention in the late nineties thorough a series of papers by Wald and Hollands (HOLLANDS; WALD, 2015), where it was realized that in order to generalize the Haag-Kastler axioms to arbitrary space-times two further axioms had to be introduced, namely Einsten causality and the time-slice axiom (FEWSTER; VERCH, 2015). The first requirement of causality reflects the necessity of the commutativity of observables localized in space-like separated regions and the second one allows one to describe the evolution between different Cauchy surfaces. The interplay of these two additional axioms with local covariance, is what induces the notion of relative Cauchy evolution and provides the theory with a dynamical structure.

Another big problem in QFT in curved spacetimes refers to the difficulty to characterize vacuum states. In QFT on Minkowski-space Poincare covariance is the requirement which allows one to single out the vacuum state, but in curved spacetimes things aren't so easy and one needs to seek a notion of invariant state on the locally covariant setting (FEWSTER; VERCH, 2015).

It is worth mentioning that QFT in curved spacetimes is in no way a description of quantum gravity, the background is classical and the quantum nature of gravity itself is assumed not to play a crucial role on the fields propagating in it.

Any attempt at a quantum description of gravity within standard perturbative QFT must face the problem of the non-renormalizability of Einstein's theory. The fact that G is a non-dimensionless physical quantity possessing dimension inversely proportional to the mass $[G] = [M^{-1}L^3T^{-2}]$ implies that the higher powers of the Riemann curvature tensor present in the perturbative series can't be supressed with a finite number of counterterms and instead yield an infinite number of divergent diagrams (KIEFER, 2012). Although non-perturbative approaches to quantize gravity have recently been undertaken such as LQG, CDT and others, these models still suffer from a number of open issues (cf. (ASHTEKAR, 1993), (KIEFER, 2006)).

3 Algebraic Quantum Field Theory

3.1 Wightman Axioms

The idea of formulating a system of axioms for QFT can be traced back to Wightman in 1964 (STREATER; WIGHTMAN, 1989) in which the primary focus was the field and quantum observables were assigned to operator-valued distributions. In essence, one reconstruct the entire theory through vacuum correlation functions, the so called Wightman functions, satisfying the axioms:

W1 $W(x_1,...,x_n) = (\psi_0, \varphi(x_1)...\varphi(x_n)\psi_0)$ are tempered distributions.
W2 (Covariance) $W(x_1,...x_n) = W(\xi_1,...\xi_{n-1}) = W(\xi) = W(\Lambda\xi), \;\; \xi_j = x_{j+1} - x_j$
W3 (Spectral Condition) The support of the Fourier transform $\tilde{W}$ of W is contained in the product of forward cones, i.e.,

$$\tilde{W}(q_1,...q_n) = 0, \text{ if, for some } j, q_j \notin \bar{V}_+ \; . \tag{3.1}$$

W4(Locality)

$$W(x_1,...,x_j,x_{j+1},...x_n) = W(x_1,...,x_{j+1},x_j,...x_n), \text{ if } (x_j - x_{j+1})^2 < 0 \; . \tag{3.2}$$

W5 (Positivity) For any terminating sequence $f = (f_0, f_1, ...f_N), f_j \in S(R^{4j})$ one has:

$$\sum_{j,k} \int dx\; dy\; \bar{f}_j(x_j,...x_1) f_k(y_1,...y_k) W(x_1,...x_j,y_1,...y_k) \geq 0 \; . \tag{3.3}$$

W6(Cluster Property) For any spacelike vector a and for $\lambda \Rightarrow \infty$:

$$W(x_1,...,x_j,x_{j+1}+\lambda a,...x_n+\lambda a) \to W(x_1,...x_j)W(x_{j+1},...x_n) \; . \tag{3.4}$$

Since the vacuum correlation functions are all that is needed for the computation of the S-Matrix elements, the Wightmann functions become the privileged objects in which the theory is built. Indeed, Wightman's reconstruction theorem asserts that in order to exhibit a relativistic quantum field theory model, it is enough to give a set of Wightman functions satisfying W1-W6. One obtains in this way a non-perturbative substitute for canonical quantization. Wightman's landmark paper then went on and further showed how one can derive both the spin-statistics theorem and the PTC theorem within this formalism (STREATER; WIGHTMAN, 1989). One thing worth pointing out though is that Haag's theorem remains a problem, though this time it manifest itself as a direct consequence of the analicity of the Wightman functions.

Theorem: Let $\phi(x,t)$ be a relativistic scalar field with vacuum state Ψ_0, satisfying:
I) At any time t is unitarily equivalent to a free field $\phi_I(x,t)$ of mass m with corresponding Fock vector $\Psi_F(t)$

$$\phi(x,t) = V^*(t)\phi_I(x,t)V(t) \ . \tag{3.5}$$

II) $\Psi_0 = V(t)^*\Psi_F(t)$, then ϕ is a free field of mass m.
Proof: We may consider a Lorentz transformation so that $x_0' = 0$ and therefore the two point function is:

$$W(y-x) = W(y'-x',0) = (\Psi_F(t), \phi_I(x',t)\phi_I(y',t)\Psi_F(t)) =$$
$$= i\Delta^+(x'-y',0;m^2) = i\Delta^+(x-y;m^2),$$

given that the propagator is Lorentz invariant. The two-point function therefore coincides with the free two-point function and spacelike points, which we can generalize to everywhere by the analicity of Wightman's functions.

3.2 Haag-Kastler Axioms

Contrary to Weightman's formalism, Haag's axioms place emphasis on the algebra of local observers. To the set of all observables belonging to a finite region O (an open set in Minkowski space) we associate an algebra $O \to A(O)$, called the algebra of observables, satisfying:

A1 (Isotony): Whenever $O_1 \subset O_2$ the corresponding local algebras are nested:

$$A(O_1) \subset A(O_2) \ . \tag{3.6}$$

A2 (Einstein Causality) If O_1 and O_2 are causally disjoint then:

$$[A(O_1), A(O_2)] = 0 \ . \tag{3.7}$$

A3 (Poincaré Covariance) To every identity connected component P_0 of the Poincaré group, there is an automorphism $\alpha(\rho)$ of $A(M)$ such that:

$$\alpha(\rho) : A(O) \to A(\rho O) \ . \tag{3.8}$$

A4 (Existence of Dynamics) If $O_1 \subset O_2$ and O_1 contains a Cauchy surface of O_2 then:

$$A(O_2) = A(O_1) \ . \tag{3.9}$$

In summary, we have that all the relevant information is contained in the net of observables which satisfies some fundamental principles of locality. We point out that it was indeed this notion of locality that motivated Haag in the late 50s, so much so that the theory was initially referred to as local quantum field theory (HAAG, 1992). The prototypical region satisfying the axioms is the double cone:

$$O = \{(t, x) \in \mathbb{R}^4 : |t| + ||x|| < l_0\} . \tag{3.10}$$

It is clear from the picture below for instance that it is casually convex:

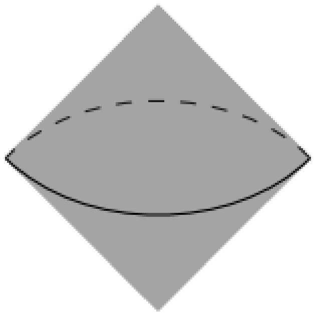


Figure 16 – Double cone region as an example of a causally convex set. (FEWSTER; REJZNER, 2019)

Furthermore, Einstein causality implies that elements of algebras of spacelike separated regions will commute, having the simple intuitive interpretation that two spacelike separated laboratories should not be able to communicate. An experimenter situated in one of the laboratories should be able to conduct experiments independently of the action of an experimenter in the other region.

The question remains however on how one can recover standard quantum mechanics in this formalism, what we shall address now. Haag's key insight stemmed from his own theorem on the inexistence of the vacuum which made evident the importance of separating the construction of observables from the construction of states. In AQFT, observables are represented by Hermitian elements of a C^*-Algebra and physical states are represented by states on this algebra. We can make this more explicit by considering the family of linear functionals $\omega : A \to \mathbb{C}$, for which we require $\omega(A^*A) \geq 0 \; \forall A \in A$ and $\omega(\mathbb{1}) = 1$. The state is mixed if it is a convex combination of distinct states and pure otherwise. Algebraic states are in fact, not so different from the ones in the usual Hilbert space setting. The theorem that allows us to connect these two concepts is no other than the famous GNS representation theorem (FEWSTER; REJZNER, 2019).

Theorem: Let ω be a state on a unital *-algebra A. Then there is a representation $(\mathcal{H}_\omega, \mathcal{D}_\omega, \pi_\omega)$ of A and a unit vector $\Omega_\omega \in \mathcal{D}_\omega$ such that $\mathcal{D}_\omega = \pi_\omega(A)\Omega_\omega$ and:

$$\omega(A) = \langle \Omega_\omega | \pi_\omega(A) \Omega_\omega \rangle, \forall A \in A . \tag{3.11}$$

What about scattering theory? First, we need to discuss how multiparticle states can exist within AQFT. Essentially one would like to create these states by acting on the vacuum through products of local operators:

$$\Psi_{1...n}(t) = A_1(t)...A_n(t)\ .\Omega \tag{3.12}$$

The key difference to stress here however, is that in QFT states can be deeply entangled, as guaranteed by the Reeh-Schlieder theorem:

Theorem: Let O be any causally convex bounded open region. Then:
I. Vectors of the form $A\Omega$ for $A \in \pi(A(O))$ are dense in H
II. if $A \in \pi(A(O))$ annihilates the vacuum, $A\Omega = 0$, then $A = 0$.

The fact that states are dense in O implies that really the entire state vector space of the field can be obtained from measurements in an arbitrarily small region of space-time. That is to say that for any two regions O_1, O_2 , no matter how far apart, there are many projections in the corresponding local algebras that are positively correlated in the vacuum state. Being more explicit, this has the effect that any experiment performed locally within a region may affect a different region of spacetime. Although this may appear to be in direct conflict to relativistic causality, it does not imply that matter and energy carried by a quantum field can travel faster than light and therefore the conflict is just apparent (VALENTE, 2014). To conclude the discussion on scattering, we point out that the existence of asymptotic states in the limit below can be shown to exist (BRUNETTI et al., 2015):

$$\lim_{t \to \pm\infty} A_1(t)...A_n(t)\Omega\ . \tag{3.13}$$

And the condition of adiabatic switching becomes somewhat less arbitrary, as it actually allows one to construct an algebra of observables associated to some bounded region, if one chooses a test function g which is equal to 1 on some larger region (BRUNETTI et al., 2015).

An interesting open problem worth mentioning is the equivalence between the Wightman and Haag-Kastler formalisms. A partial answer to this problem was obtained by Borchers and Yngvason at (BORCHERS; YNGVASON, 1992) where it was shown that the necessary and sufficient conditions for the existence of a local net of von Neumann algebras corresponding to a given Wightman field can be formulated in terms of strengthened versions of the positivity property (W5 in the text above). In essence, if the smeared field operators are self-adjoint for real test functions and commute strongly, one can define a local Von Neumann algebra $M(O)$ generated by the bounded functions of the field operators:

$$M(O) = \{F(\phi(f))|F \text{ bounded, support } f \subset O\}\ . \tag{3.14}$$

3.2.1 Models of Interest

To illustrate the construction of models in AQFT we shall start with the simplest case of the free scalar field:

$$(\Box + m^2)\phi = 0 \ . \tag{3.15}$$

Denoting by E the Green function propagator $\phi = Ef$ which solves the inhomogeneous equation:

$$(\Box + m^2)\phi = f, \tag{3.16}$$

we may write:

$$E(f,g) = \int_M f(x)(Eg)(x)d^4x, \tag{3.17}$$

and the integral kernel will be given by:

$$E(x,y) = -\int \frac{d^3k}{(2\pi)^3} \frac{\sin k(x-y)}{\sqrt{||k^2|| + m^2}} \ . \tag{3.18}$$

The quasi-local algebra A(M) in this case can be constructed by the distributions $\phi(f)$ where the test functions $f, g \in C_0^\infty(M)$ satisfy:

P1 (Linearity) $f \to \phi(f)$ is complex linear.
P2 (Hermiticity) $\phi(f)^* = \phi(\bar{f})$.
P3 (Field equation) $\phi((\Box + m^2)f) = 0$.
P4 (Covariant commutation relations) $[\phi(f), \phi(g)] = iE(f,g)\mathbb{1}$.

Imposing the axioms above is equivalent to Dirac's quantization procedure. Next, we shall define for each causally convex open bounded $O \subset M$ the algebra $A(O)$ generated by elements $\phi(f)$ for $f \in C_0^\infty(O)$ along with the unit $\mathbb{1}$. Therefore, it is clear that if $O_1 \subset O_2$ we have $A(O_1) \subset A(O_2)$ and it follows that axioms A1 and A2 are satisfied. Einstein causality arises as a consequence of E(f,g)=0, when f and g have causally disjoint support. Poincaré covariance is a manifest consequence of the covariance of E and the existance of dynamics can be shown as follows.

Theorem: Let $O_1 \subset O_2$ such that O_1 contains a Cauchy surface of O_2. Then any solution $\phi = Ef_2$ for $f_2 \in C_0^\infty(O_2)$ can be written as $\phi = Ef_1$ for some $f_1 \in C_0^\infty(O_1)$.

Proof: Take $f_1 = P\chi\phi$ where $\chi \in C^\infty(O_2)$ vanishes to the future of one Cauchy surface of O_2 contained in O_1 and equals unity to the past of another. Then $\phi(f_2) = \phi(f_1)$, which implies $A(O_2) = A(O_1)$.

Now we'll consider a slightly different example, the case of a scalar field with external source, the so-called Van Hove model:

$$(\Box + m^2)\phi_\rho = -\rho \ . \tag{3.19}$$

We define the smeared fields as:

$$\phi_\rho(f) = \phi(f) + \phi_\rho(f)\mathbb{1} \ . \tag{3.20}$$

The relations that must be imposed now are:

P1 $f \to \phi_\rho(f)$ is complex linear.
P2 $\phi_\rho(f)^* = \phi_\rho(\bar{f})$.
P3 $\phi_\rho((\Box + m^2)f) + \rho(f)\mathbb{1} = 0$.
P4 $[\phi_\rho(f), \phi_\rho(g)] = iE(f,g)\mathbb{1}$.
it is more favourable now to consider the Weyl algebra satisfying:
P1 $W(\phi)^* = W(-\phi)$.
P2 $W(\phi)W(\phi') = e^{-i\sigma(\phi,\phi')/2}W(\phi + \phi')$.

to construct the quasi-local algebra. We note that this is valid, given that:

$$(e^{i\phi(f)})^* = e^{-i\phi(f)} = e^{i\phi(-f)}, \tag{3.21}$$

and:

$$e^{i\phi(f)}e^{i\phi(g)} = e^{i\phi(f)+i\phi(g)-[\phi(f),\phi(g)]/2} = e^{-iE(f,g)/2}e^{i\phi(f+g)} \ . \tag{3.22}$$

We can then go on and construct local algebras by defining $W(O)$ as the C^*-subalgebra generated by $W([f])'$s with supp$f \subset O$ and O being any causally convex open bounded subset of M. It can be shown analogously that this algebra satisfies the axioms A1-A4 of AQFT (FEWSTER; REJZNER, 2019).

Lastly, we shall consider an interacting theory. In particular, we are interested in one of the very first models constructed by Glimm and Jaffe in the late 60s, the two-dimensional ϕ^4 model (GLIMM; JAFFE, 1968). Consider the cut-off interacting Hamiltonian operator:

$$H(g) = H_0 + \lambda \int :\phi(0,x)^4: g(x)\, dx \ . \tag{3.23}$$

The free scalar hermitian Bose field $\phi(t,x)$ and the canonically conjugate momentum field $\pi(t,x) = \partial\phi(t,x)/\partial t$ satisfy the commutation relations:

$$\phi_0(f)\pi_0(g) - \pi_0(g)\phi_0(f) = i\langle f, g\rangle\mathbb{1}, \tag{3.24}$$

$$\phi_0(f)\phi_0(g) - \phi_0(g)\phi_0(f) = \pi_0(f)\pi_0(g) - \pi_0(g)\pi_0(f) = 0, \tag{3.25}$$

and once again our algebra of observables will be generated by the set of Weyl unitaries:

$$\{e^{i\phi_0(f)}, e^{i\pi_0(f)} | f \in L^2(\mathbb{R}), \text{supp}(f) \subset O\} \ . \tag{3.26}$$

For any $t \in \mathbb{R}$ let O_t denote the subset of $\mathbb{R}$ consisting of all points with distance less than $|t|$ to O. By choosing the cutoff function g to be equal to $\mathbb{1}$ on O_t, then for any $A \in A(O)$ the operator:

$$\sigma_t(A) = e^{itH(g)} A e^{-itH(g)}, \tag{3.27}$$

is independent of g and is contained in $A(O_t)$. For any bounded open $O \in \mathbb{R}^2$ and $t \in \mathbb{R}$, let $O(t) = \{x \in \mathbb{R} | (t, x) \in O\}$ be the time t slice of O. We define A(O) to be the von Neumann algebra generated by $\cup_s \sigma_s(A(O(s)))$. This illustrates the procedure alluded before on how the adiabatic limit may induce a net of local algebras. This formalism can then be shown to satisfy both Haag-Kastler and Wightman axioms (SUMMERS, 2012).

The construction of a three-dimensional ϕ^4 interacting theory is also possible and has been carried out by Glimm and Jaffe but for dimensions $d \geq 4$ we inevitably run into the issue of renormalizability. To put shortly, the ϕ^4_d model is superrenormalizable in $d \leq 3$, whereas in d=4 it is only renormalizable, and in $d \geq 5$ it is nonrenormalizable, which makes the construction difficult. To be more specific, it has been shown that the Glimm and Jaffe procedure carried out in two and three dimensions breaks down in 4D and one is left with a free non-interacting theory. (SUMMERS, 2012)

3.2.2 Challenges in Algebraic Gauge Quantum Field Theory

The main issue with formulating gauge field theories in AQFT has to do with locality. A gauge QFT consists of conservation laws and charges Q^α which commute with all the observables, so that the Hilbert space is decomposed into a direct sum of spaces H_q consisting of charged states:

$$H_0 = \overline{\{A\psi_0\}} \; .$$

These are called the superselection sectors of the theory. The problem arises when one considers the consequences of Gauss' law for local gauge groups:

$$Q^\alpha = \int d^3x j^\alpha_0(x,t),$$

one not only has the conservation rule associated to the conservation law $[Q^\alpha, H] = 0$, but also a different commutator $[Q^\alpha, A]$, for any local observable. As a matter of fact, a charge carrying field possess the property:

$$[Q, \phi] = q\phi \; .$$

On the other hand,by Gauss' law:

$$[Q, \phi(y)] = \int d^3x [j_0(x, 0), \phi(y)] = [\Phi_\infty(E), \phi(y)],$$

where $\Phi_\infty(E)$ denotes the flux at spacelike infinity of the field. If the quantities commute at spacelike separations the right hand side should be zero, but on the other hand $\phi(y)$

cannot have a non zero charge Q at infinity. Gauss' law therefore implies a delocalization of charged states, which goes against the principles of AQFT. To the present day, it has been shown by Doplicher, Haag and Roberts that observables in a global gauge theory can lead both to the gauge group and to charge carrying fields (HAAG, 1992) in agreement to the axioms of AQFT, however an equivalent result for local gauge fields is yet to be found, given the reasons presented above.

3.3 pAQFT, CFT and Beyond

There have been a number of major developments in AQFT in the past few years. One solution to the problem of gauge theory that has been given attention recently is Perturbative Algebraic Quantum Field Theory. The axioms within this formalism are rephrased in terms of locally convex topological vector spaces with some additional structure. A classical field theory, on a spacetime M, is given by a net of locally convex topological Poisson *-algebras $B(O)$, each with sequentially continuous product and a Poisson bracket to enforce locality.

$$\lfloor A, B \rfloor = 0 \; .$$

Moreover, the spectral condition is rephrased in terms of Hadamard states and the kinematics are derived within the usual Lagrangian formalism but with some modifications to the configuration space, namely the introduction of the Peierls bracket. (BRUNETTI et al., 2015)

Conformal field theories have also been studied within AQFT. The Haag-Kastler axioms are shown to hold just fine for this case, having no need to introduce additional "CFT axioms". One needs however a way to translate the conformal symmetry of the theory as a consequence of the AQFT axioms. One model in particular that has been thoroughly investigated is the case of CFT on the circle, it has been shown that a Möbius covariant chiral QFT with a net of local algebras $A(I)$ defined over intervals on the circle S^1 is in agreement with the AQFT formalism. The possibility of holographic ideas such as AdS/CFT to construct models is also something currently being investigated. (BRUNETTI et al., 2015).

All of these developments, together with the program initiated by Wald and Hollands on AQFT on curved spacetimes (HOLLANDS; WALD, 2015) form a topic of intensive research today.

Conclusion

In the first part of this thesis we've laid out the main problems with the standard perturbative approach in quantum field theory. We then proceeded to show how algebraic quantum field theory addresses some of these problems and explored the construction of a few simple models. We abstained from providing too much detail and touching on complicated topics such as AQFT in curved spacetimes and gauge theories within the AQFT formalism as this was not the primary objective of this work. Any introduction to such a vast topic as is AQFT is bound to leave a few many things behind, but it is our hope that we managed to provide some intuition to some of the key ideas and concepts in this field of great importance to theoretical physics. As Freeman Dyson once said (DYSON, 1972):

"These axioms, taken together with the axioms defining a C^*-algebra, are a distillation into abstract mathematical language of all the general truths that we have learned about the physics of microscopic systems during the last 50 years. They describe a mathematical structure of great elegance whose properties correspond in many respects to the facts of experimental physics. In some sense, the axioms represent the most serious attempt that has yet been made to define precisely what physicists mean by the words observability, causality, locality, relativistic invariance, which they are constantly using or abusing in their everyday speech."

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